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Logical graphs – how to map mathematics

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Abstract: A logical graph is a certain directed graph with which any mathematical theory or proof can be presented – its logic is formulated in graph form. Compared to the usual verbal "narrative" description, the presentation usually gains in survey, clarity and precision.

A logical graph formulation can be thought of as a detailed and complete map over the mathmatical landscape. The main goal in the design of logical graphs is didactical: to improve the orientation in a mathematical proof or theory for a reader, and thus to improve the access of mathematics.

Kurzreferat: Ein logischer Graph ist ein bestimmter gerichteter Graph, durch den eine mathematische Theorie oder ein mathematischer Beweis dargestellt werden kann. Im Vergleich zu der üblichen verbalen, „erzählenden" Beschreibung, gewinnt diese Darstellung in der Regel an Übersicht, Klarheit und Präzision.

Die Formulierung mithilfe eines logischen Graphen kann als detaillierte und vollständige Abbildung über der mathematischen Landschaft aufgefaßt werden. Das Hauptziel, logische Graphen zu entwickeln, ist ein didaktisches: die Orientierung in einem mathematischen Beweis oder einer mathematischen Theorie zu verbessern und damit auch den Zugang zur Mathematik.

ZDM-Classification: E30, E40

"We have not begun to understand the relationship between combinatorics and conceptual mathematics."
J. Dieudonné, *A panorama of pure mathematics* (1982)

A logical graph is a way of notating classical logic as a directed graph, which allows a graphical presentation of any mathematical proof or theory. In logical graph form the reader obtains an immediate view over the logic, without losing any detail information. The writer has much freedom in the design of logical graphs – for example to choose detail exposition – in order to obtain maximum clarity of presentation.

We first give the formal definitions, and then discuss their use to provide easier understanding of mathematics.

1. Definitions

Definition 1: A logical graph is a directed graph with nodes of two kinds, called propositions and definitions, and edges of two kinds, implications and constructions. Each proposition node contains at least one statement, new notation may also be introduced simultaneously. Each proposition node A represents the statement $A_1 \wedge ... \wedge A_k \Rightarrow A$, where $A_1, ..., A_k$ are all implication predecessors of A.

Definition nodes contain no statements, but introduces new notation. A construction edge from node B, usually a definition node, to the node C denotes that in C some notation defined in B is explicitly used. No implication edge ends at a definition node, since we never prove definitions. (See all examples)

We need a few more formal tools to justify the claim that logical graphs can be used to present any proof. Sometimes we make assumptions within a proof. One such case is proof by contradiction. Another case is when we take care of different cases with different proof methods. This calls for a few more definitions, of which the first is the following:

Definition 2: A logical subgraph is a subgraph of a logical graph having one or more assumptions stated in one entrance proposition node. An implication may only cross the boundary of a subgraph from the outer side to the inner side. (See Examples 1 and 3)

Definition 3: A proof by contradiction is a subgraph of a logical graph with one entrance proposition node A, the assumption, and one exit proposition node False, the contradiction. Thus A is established. With a slight abuse of notation this is denoted as implications from the entrance A and from the exit False, as if A⌈False⌋— A. Inferences may point from outside the proof by contradiction to a proposition within, but no implication may point in the reverse direction. (See Example 3)

A proof by cases may consist of one subgraph for each case. Suppose that we want to prove A, and make assumptions $C_1, ..., C_k$ where $C_1 \vee ... \vee C_k \Leftrightarrow$ True. Here we obtain k subgraphs each with entrance node C_i and exit node A. We thus prove $C_i \Rightarrow A$ for all i: $1 \leq i \leq k$, and A follows.

Every case does not need to be written out as a subgraph. In the proof of the Hahn-Banach theorem, the case $\| g \| = 1$ contains the main difficulties. The case $\| g \| = 0$ is simple enough to be treated in one proposition node only, and perhaps this is appropriate also for the case $\| g \| > 0$, $\| g \| \neq 1$. Any presentation method should provide easy notation for common and useful ways of argument; it is important not to limit the freedom of expression unnecessarily. The main goal of logical graphs is clarity and easy expression.

Inferences are not allowed between subgraphs, since we in different subgraphs have different sets of assumptions. Furthermore it is clearly possible to have subgraphs in a subgraph. By defining the inner side of the boundary in the natural way, we may summarize this in a border crossing rule, which is part of the definition of a logical subgraph:

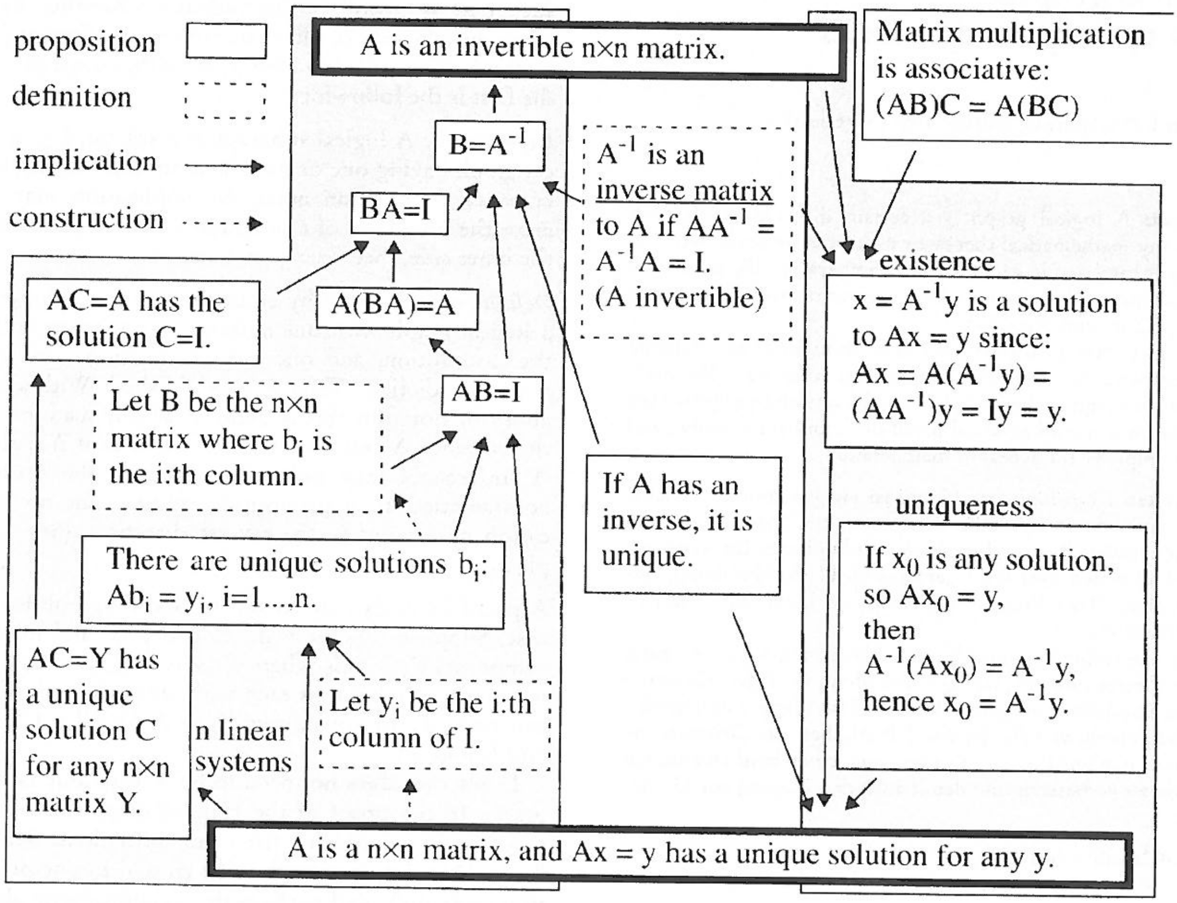
Border crossing rule: An implication may cross any number of boundaries, but every crossing must take place from the outer side to the inner side.

2. Comments on the definitions

"and" and/or "or". Definition 1 is enough to present a theory. Note that, since implication edges together form an implication statement, all implication edges ending at the same proposition should be viewed as a group.

The "and" and "or" operations can easily be denoted as the following example 5. However, the logical connections in proofs and theories are usually formulated with the operation "and"; we have a set of assumptions, and if all are true a resulting statement is also true. Therefore "and" is chosen as standard notation.

Theory graphs and proof graphs. When used to present a theory, the proposition nodes are lemmas and theorems. Axioms are in principle definitions with no incoming constructions. However, explanations of the axioms may require more primitive definitions. For



Example 1. Displayed proof: If A is a square matrix, A is invertible if and only if Ax=y has a unique solution x for every y.

example, one may argue that the axioms in group theory relies on the function concept.

When used to present a mathematical proof, the nodes are steps in the proof, chosen as usual by appropriate degree of explicity. Of course, the basic rule here is the one of Aristotle: each step should be evident. When used to present proofs, there is a set of proposition nodes which is special: the assumptions – these have no incoming implications. Another special set is the set of conclusions. It is recommended to make these special proposition nodes extra visible, e.g. by bold frames. In definition nodes it is further recommendable to underline new notation and new terms, especially in the case of theory graphs.

Example 1 clearly presents two proofs, since the theorem states an "if and only if"-result; the role of being premise and conclusion is interchanged in the two proofs. The direction of the arrows makes it clear what is premise and what is conclusion.

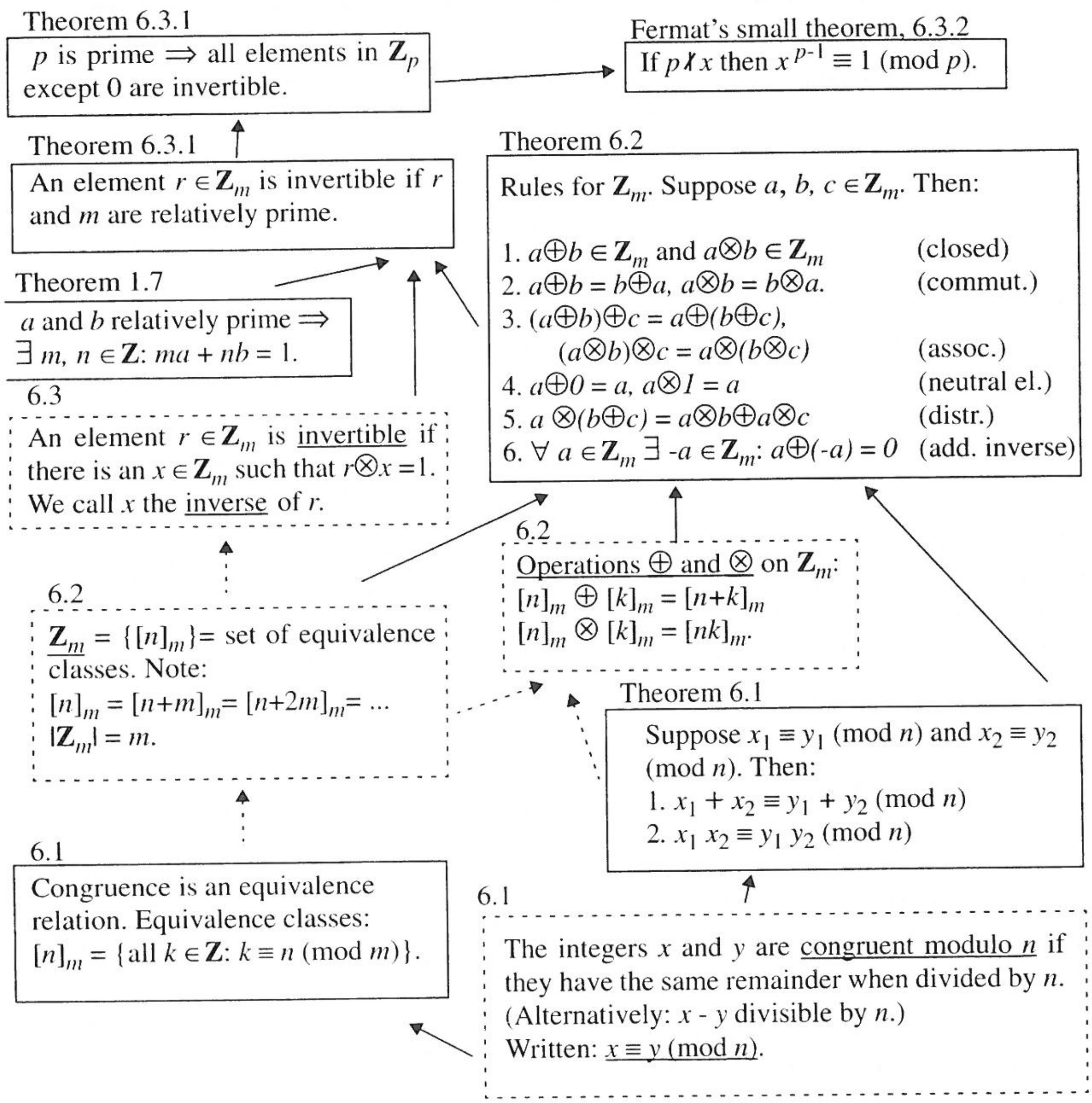
Goal: maximal clarity. The goal of logical graphs is not to atomize arguments as far as possible, to make everything into a graph which can be transformed in this way. The overall goal is clarity of argument, which sometimes is served best by allowing several components, i.e. propositions and definitions, in the same proposition node. This does not disturb the logical structure, since if P and Q are propositions, also $P \wedge Q$ is of course a proposition. The constructor has a lot of freedom in logical graph design, just as in narrative proofs. What is evident and what is not evident is of course very dependent on the audience. As with narrative presentation, with logical graphs one can freely choose any degree of detail exposure.

What is statement, what is definition? In the examples provided here one can observe that the distinction between definitions and propositions is not completely sharp; the dominating character is indicated in the graphs. It is sometimes questionable whether a certain definition is possible, whether a well defined object is defined. If so, it is also a proposition. Conversely, propositions often naturally introduce new notation. One example is the verb "take" which is the proposition "there exists", but is usually intended to introduce a name for a mathematical object. If the existence is at stake, such a node should be a proposition node.

Earlier matter. Earlier results or definitions which are needed in a proof or included as reminders appear with one side missing. There are some examples of this in the graphs in this text.

Hypertext. Partitioning a proof in subproofs suggests the use of hypertext. The reader could then click on a difficult step and obtain a subgraph displaying the arguments by degrees with more details and more explanation.

Relation between definitions. Constructions are certainly a part of mathematics. If notation from definition D_1 is used in definition D_2, there is clearly a specific formal relationship between the two definitions. However, there is no commonplace mathematical notation for this



Example 2. Displayed theory: Basics of modular arithmetic. Numbering is according to corresponding course book.

relationship. It is interesting to note how a different but equivalent formal representation bring different ideas to the surface. This is of course an observation about how the human mind works.

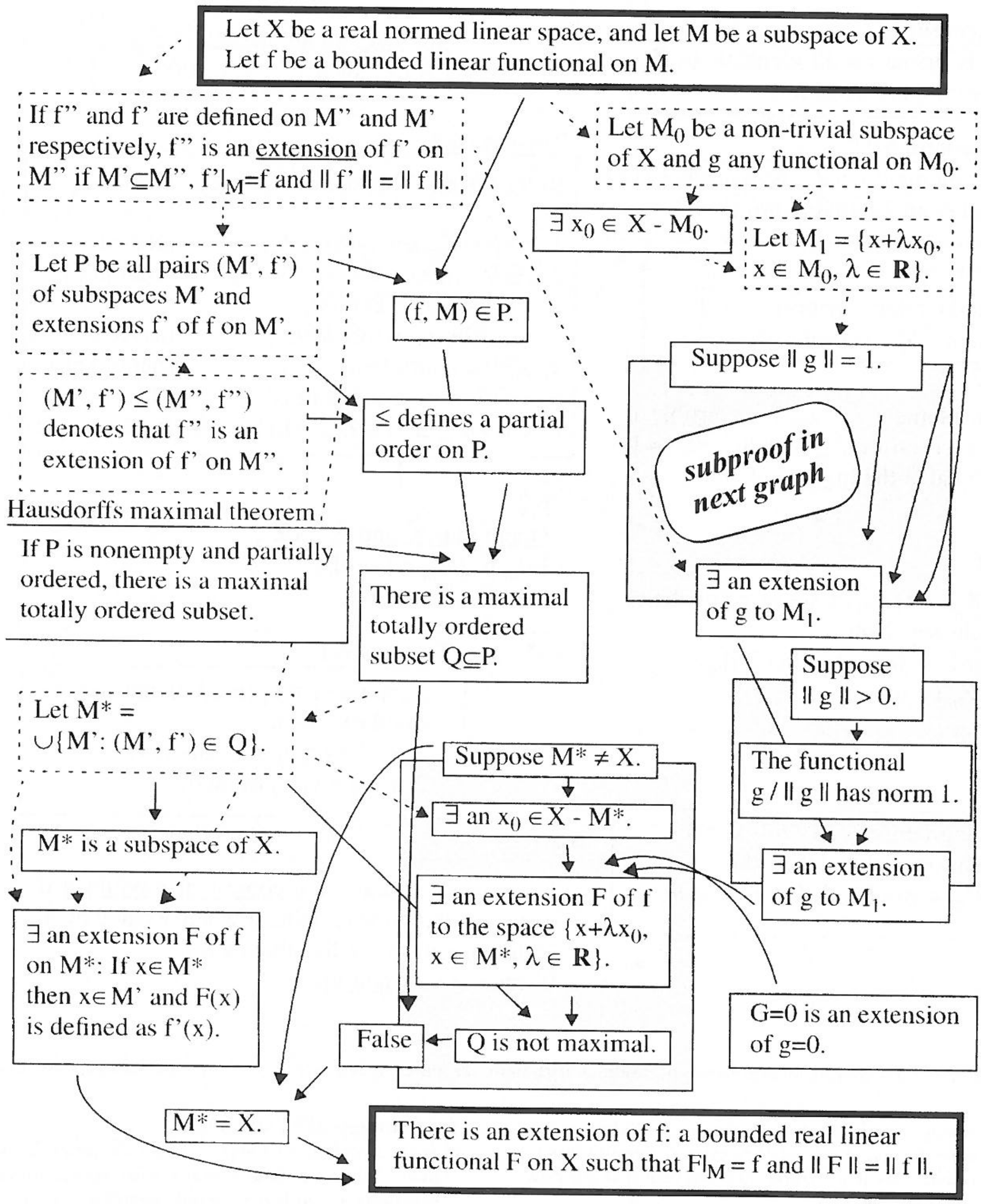
Proofs are graphlike. We all know that the sequence of implications from assumption to result rarely is straight and branchfree like a flag pole, it is rather like a bush or a jungle. A graph is a very natural concept to catch this reality.

Do we need city maps? On the other hand, a lecturer has to present a theory linearly, since the time is linear, at least locally. One could then ask: should mathematical texts be written lectures, reflecting the shape of time, or should written presentations of mathematics reflect the structure of its content? A similar question is: Do we really need city maps? Would we find the right spot in a city easier if we instead use a collection of purely verbal tour descriptions?

3. Advantages of logical graphs

1) *More complete.* In a logical graph all "hence", "it follows" "thus we obtain" and similar synonymous expressions, far to few to avoid repetition, are replaced by implication arrows. In a narrative text we cannot explicitly mention all implications. Doing this every time would lead to a very heavy account, in fact obstructing the content. The clumsiness of natural language, for mathematical purposes, forces narratively presented proofs to be rather incomplete. As described, in logical graph form one is free to keep any degree of incompleteness, whatever serves clarity.

2) *More liberty for the reader.* When reading a proof presented by a logical graph, the reader is free to choose if to start from the hypothesis and contemplate the starting reformulations, or to start in the other end, from the result. In a narratively presented proof, instead the writer decides in which order to read the proof. Certainly, after having read through a



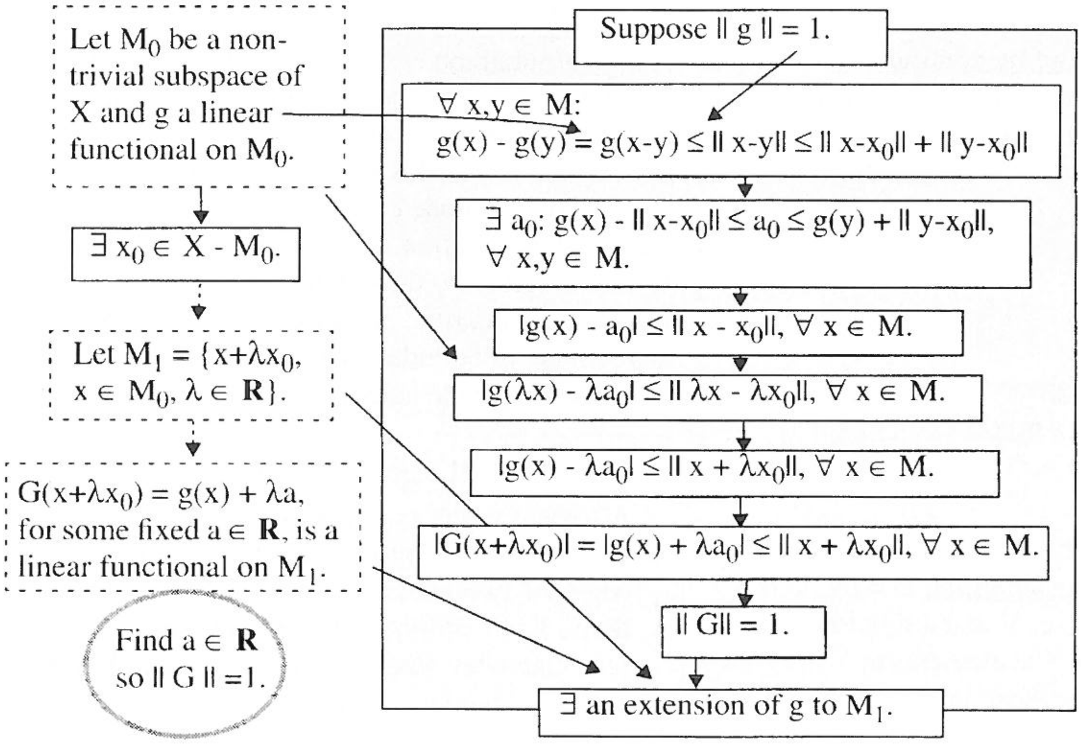
Example 3: Proof of the Hahn-Banach theorem, a fundamental result in functional analysis.

few times, the reader may be free to read backwards, or to reconstruct the proof in another order. A consequence of this and of advantage No. 1 above is that the reader can faster separate the trivialities from the difficulties, and concentrate upon real problems. Because of the visibility of the overall logical structure, also the possibility of grasping a proof as a single idea comes within closer reach.

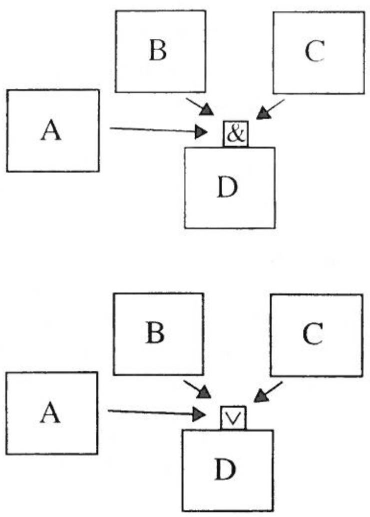
3) *More conceptually adequate.* I claim that a picture of this kind is really what one tries to construct in the head when studying a mathematical theory.

4) *More independent of natural language.* Logical graphs make it possible to present mathematical research reports with very rudimentary natural language. Mathematics takes then one step closer towards being a truly international language, though a language for a special purpose. Doors may open for mathematicians who do not understand english, as is often the case in developing countries. However, reports completely free from natural language can hardly be recommended, a supplementary text should always join the graph. Natural language is required for several reasons – one is to avoid disadvantage no 2 below.

5) *Results first.* If possible it is preferable to read a mathematics course backwards: to start with the re-



Example 4. Remaining subproof of the Hahn-Banach theorem.



Example 5. Alternative notation for the propositions $A \& B \& C \Rightarrow D$ and $A \vee B \vee C \Rightarrow D$, respectively.

sults and goals, and later tend to the construction of the theory, being aware of the final results. This is easier to carry through in logical graph form, partly because of the constructions; it is easier to track down definitions of unknown concepts to their well known origins. It is an easier task to separate the unknown from the known parts of the theory.

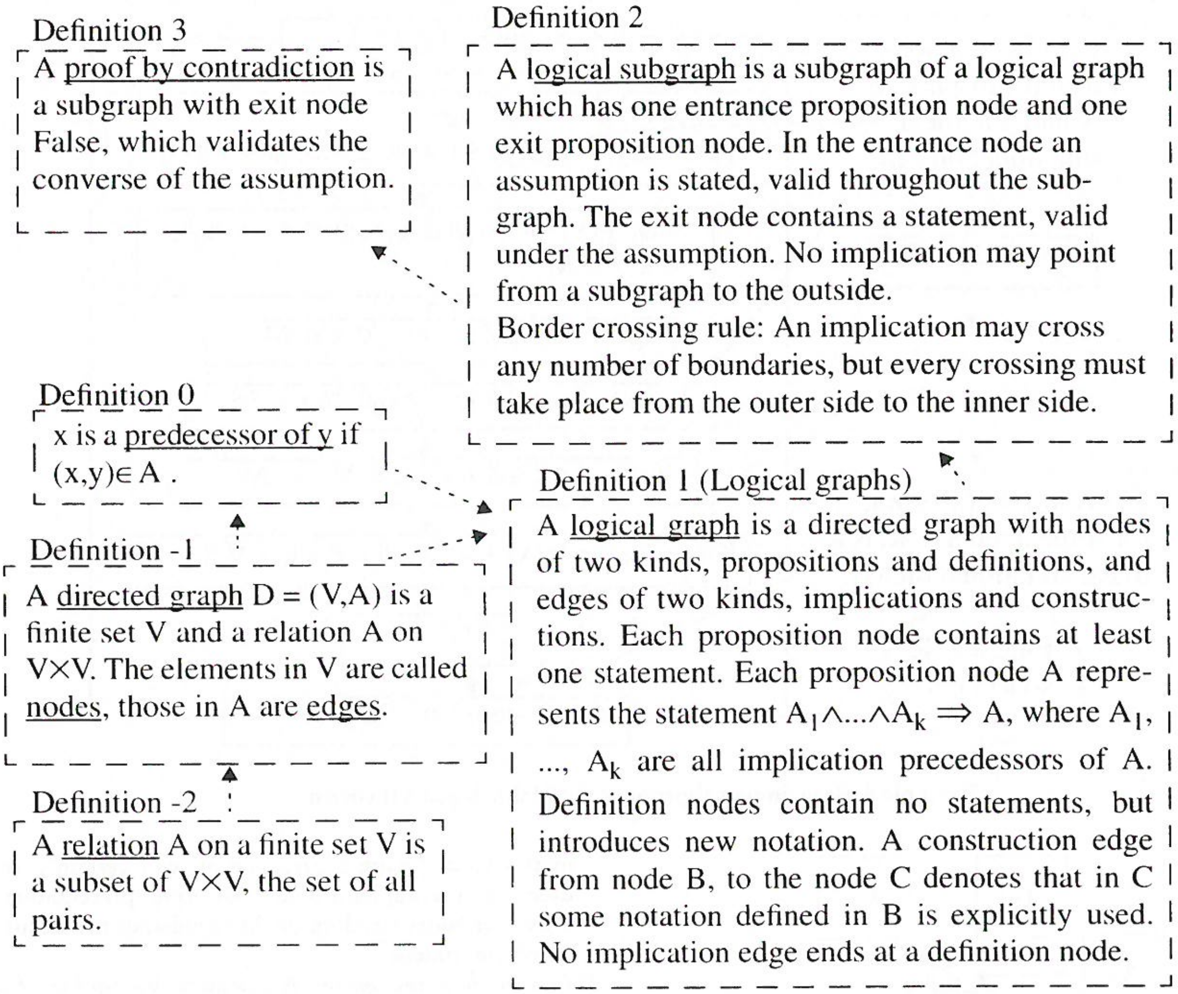
4. Disadvantages of logical graphs

1) *Demands more of the reader.* The advantage of more freedom may also be a disadvantage. The reader must make more decisions, e.g. where to start reading. Logical graphs offer a weaker leadership on the mathematics tour. Graphic presentation can certainly be used as a complement to narrative presentation. However, more freedom ought to enhance the maturity of the student.
2) *Less method description.* A narrative description of a proof may contain information of mathematical methods and ideas which have no formal significance, but which is important for how to construct and derive such results. The ultimate motivation for a certain definition is of course that it works. How to choose definitions and find proof methods is a matter of analogy, creativity, intuition; it is a matter "above" strict mathematics. It should however be possible to deliver such information in commentary nodes or in supplementary text.
3) *Lack of software,* making the design of logical graphs easy. The logical graphs in this article have been designed with Framemaker 4.0.

5. Logical structure consistent with intution

Intuition, logic, learning. Certainly, a lot can be gained for the future of mathematics if students easier grasp our courses and researchers easier grasp new results. Hard work will of course always be needed, with any presentation method. However, with a presentation which is consistent with intuition, it may be faster and more rewarding. The main idea of logical graphs can be described very shortly as a certain kind of laziness – to avoid to restate a proposition or definition. When it reappears, we instead draw an arrow from it – a graph results. At the same time the logical relationship can be indicated.

Nodes for comments, graphics, applications... Logical graphs can further be completed with more structure for comments, such as indications of proof ideas, intuitive



Example 6. Logical graphs as form and content.

connections, examples, geometrical plots and so on. The reader can find some of such extensions in the provided examples. Any further structure will however decrease the visibility of the logical structure, so it should be well motivated. Without doubt, graphic display of mathematical ideas occur and has occurred several times at various instances and in various forms. This is an attempt to define natural, reliable and generally usable basic concepts. The author has displayed about 15 graduate courses in mathematics by logical graphs without problems.

6. A few words on logical graphs in teaching

Map of textbook. If logical graphs are made in conjunction with a specific textbook, enumeration and page numbering can follow each theorem and lemma node. This gives a map to the book, results and conceptual connections are easier to find, as well as the structure of the content.

What remains is understanding. Allowing theory graphs at exams focuses on understanding. The formulation of definitions and theorems is then provided, as well as implications, i.e. information about which results should be used in the proof. What is not provided is proofs and solutions to problems – then the natural subject of the exam. This extra help for the students implies the possibility to include a few more topics in the course.

Decode formalism. A condition for the student to be able to use the theory graphs is clearly to understand the mathematical formalism. The task for the student during the course can then be described as to decode the theory graphs, and lectures can attempt to bridge intuition and formalism. Logical graphs probably also help memorizing – it is well known that detail knowledge survives better once the role of details in an overall picture is understood.